

Narrow rings: a scattering billiard model

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We propose a generic mechanism for the formation of narrow rings in rotating systems. For this purpose we use a system of discs rotating about a common center lying well outside the discs. A discussion of this system shows that narrow rings occur, if we assume non-interacting particles. A saddle-center bifurcation is responsible for the relevant appearance of elliptic regions in phase space, that will generally assume ring shapes in the synodic frame, which will suffer a precession in the sidereal frame. Finally we discuss possible applications of this mechanism and find that it may be relevant for planetary rings as well as for semi-classical considerations.

§1. Introduction

With the discovery of ring structures around all mayor planets ¹⁾ the phenomenon has past from being a breathtaking anomaly in the sky to a rather common feature. While there exists an extended body of work studying features, origin and destiny of planetary rings, ^{1) - 5)} we believe that there is room for a more general question: How generic is the formation of narrow rings? We shall here discuss a model which is quite remote from any true representation of planetary rings. The purpose will be to show that very simple rotating systems can and usually will have a phase space structure that allows for the occurrence of rings, even if the particles of such a ring do not move on trajectories that could be approximated by the rings shape. Understanding the mechanism in a simple model should reveal the genericity we seek.

We shall start by generalizing the model of rotating discs which we proposed earlier. ⁶⁾ In section 3 we discuss the stability of periodic orbits in this system and the ring structures we find as a consequence of stable motion for particles not interacting among themselves. Next, we shall see how the introduction of a second disc on a smaller orbit may limit the number of rings to as few as a single one. In particular we shall see that we do not require both discs to move at the same angular velocity to maintain the ring structure. We proceed finally to consider the genericity of the results obtained in this particular model. ⁷⁾ In particular, we shall discuss the consequences of using smooth potentials and even attractive ones such as the $1/r$ potential that occurs in the planetary rings as well as possible applications.

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§2. The rotating two-disc scattering billiard

We shall study the planar motion of collisionless point particles in an open billiard which consists of two non-overlapping discs on circular orbits (Fig. 1). We consider discs of radii d_i whose center are at distances R_i from the origin ($i = 1, 2$). For simplicity, we shall discuss the case where the discs move with the same angular velocity ω , which permits a time independent Hamiltonian formulation, although the dynamics of the billiard will allow us to relax this condition. The (constant) angle formed by the vectors pointing to the centers shall be denoted by β . In what follows, we shall refer to the inner and the outer discs as disc 1 and disc 2, respectively. The non-overlapping condition reads as $R_1 + d_1 < R_2 - d_2$.

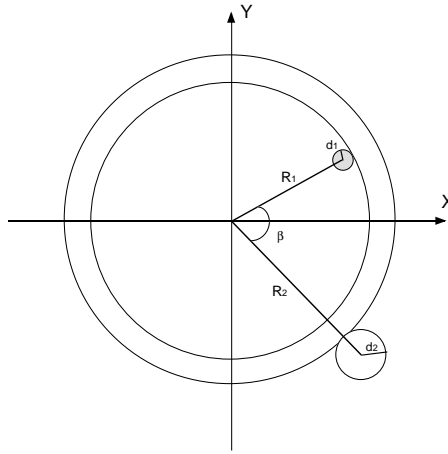


Fig. 1. Geometry of the two rotating discs scattering model: R_i ($i = 1, 2$) define the position of the centers of the discs and d_i their radii; β defines the relative positions of the discs.

In a rotating frame (synodic frame), the dynamics are given by the Hamiltonian

$$J = \frac{1}{2}(p_x^2 + p_y^2) - \omega(xp_y - yp_x) + V(x, y) \quad (2.1)$$

with $V(x, y) = 0$ outside the discs, and infinite inside. The explicit time independence of J in Eq. 2.1 implies that it is a conserved quantity of the global flow, which is known as the Jacobi integral. We shall define the coordinates such that in this rotating frame the outer disc lies on the positive x -axis. We note that, if the discs move with different angular velocities, there is no first integral of motion.

On a fixed frame (sidereal frame), the dynamics are straightforward. A particle moves on a straight line with constant velocity until it hits one disc, which changes both the magnitude and the direction of the velocity vector.⁶⁾ After the collision, a point particle wins (loses) energy if it bounces off the discs on their front (back) side. This naive observation is precisely the key point for the construction of the symmetric periodic orbits of the system. If a particle bounces radially, that is at one of the intersection points of the disc and the line that joins the center of rotation and the center of the disc, then the energy is conserved and incoming collision angles

are equal to outgoing ones. If we have initial conditions which display consecutive radial collisions with the disc, they belong to a trapped orbit. In the synodic frame, such trapped orbits are actually periodic orbits which preserve the symmetry of the problem.⁶⁾ As shown in Ref. 8), the fundamental periodic orbits in the construction of the chaotic saddle are precisely the symmetric periodic orbits, or deformations of them for the non-symmetric cases.

We shall first consider either of the two discs by itself. In terms of the Jacobi integral J_{ii} ($i = 1, 2$), their symmetric periodic orbits are given by⁶⁾

$$J_{ii} = \frac{\omega^2(R_i - d_i)^2}{[(2n + 1)\pi - 2\alpha_i]^2} \left[2 \cos^2 \alpha_i - ((2n + 1)\pi - 2\alpha_i) \sin 2\alpha_i \right]. \quad (2.2)$$

Here, $\alpha_i \in [-\pi/2, \pi/2]$ is the outgoing angle for the radial collision with the disc (negative values correspond to a particle running against the rotation), and $n = 0, 1, 2, \dots$ denotes the number of full turns that the disc completes before the next collision. These curves are plotted for some values of n in Fig. 2a as continuous lines. Notice that n defines continuous families of such periodic orbits with a precise hierarchical arrangement.

Similar families of periodic orbits exist which alternate bounces among the discs. However, symmetry considerations of these two-bounce periodic orbits imply that they retain the symmetric character, i.e. bounce radially, only for $\beta = 0$ and $\beta = \pi$. That is, except for those values of β , related periodic orbits are alternatively accelerated and decelerated after collisions with the discs. The two-bounce symmetric periodic orbits are given by

$$J_{12} = \frac{1}{2}v_p^2 - \omega(R_2 - d_2)v_p \sin \alpha_2 \quad (2.3)$$

where the (constant) velocity between collisions is (for $\beta = 0, \pi$)

$$v_p = \frac{\omega(R_2 - d_2)}{[n\pi + \beta + \alpha_1 - \alpha_2]} \frac{\sin(\alpha_1 - \alpha_2)}{\sin \alpha_1}. \quad (2.4)$$

In these cases α_1 and α_2 are geometrically related by the identity

$$\frac{\sin \alpha_2}{R_1 + d_1} = \frac{\sin \alpha_1}{R_2 - d_2}. \quad (2.5)$$

Notice that Eq. (2.5) defines the limits $|\alpha_2| \leq \alpha_2^{\max}$ for which the orbits J_{12} exist, where $\sin(\alpha_2^{\max}) = (R_1 + d_1)/(R_2 - d_2)$. The case where the equality holds correspond to tangent collisions with disc 1. Again, n is defined as the number of turns completed by the discs during one period of the symmetric periodic orbit.

In Fig. 2b we have plotted the symmetric periodic orbits which involve disc 2 for $\beta = 0$ for some values of n , as given directly by Eqs. 2.2 and (2.3-2.4). It follows from Eqs. (2.3-2.4) that for $\beta = \pi$ the n curves correspond to the $n + 1$ curves of $\beta = 0$. As it can be appreciated in Fig. 2b, the curves J_{12} display cusps at the edges of the interval where they are defined ($|\alpha_2| = \alpha_2^{\max}$) and decrease monotonically.

We notice that the curves J_{22} (continuous lines in Fig. 2a) are convex or concave depending only on the sign of α_2 , and then display local maxima and minima. The

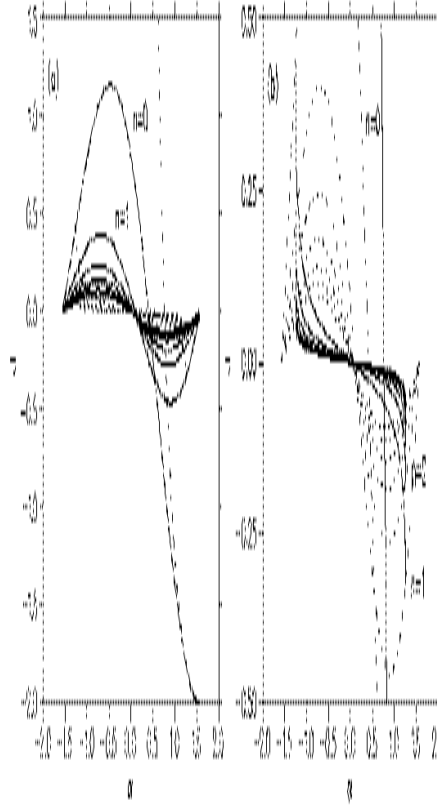


Fig. 2. (a) Periodic orbits J_{22} (continuous curves) and J_{12} for $\beta = 0$ (dotted curves) for various values of n ($R_1 = 1.8$, $d_1 = 0.1$, $R_2 = 3$, $d_2 = 1$). Notice the divergence of J_{12} as $\alpha \rightarrow 0$ for the $n = 0$ case. (b) Enlargement of (a) showing the periodic orbits of J_{12} for $\beta = 0$ (continuous curves) for various values of n ; some periodic orbits J_{22} are shown as dotted curves. The family of periodic orbits for $\beta = \pi$ are obtained from those for $\beta = 0$ by rearranging the labels (see text).

important fact to observe here is that the derivative of the J_{12} families is strictly negative for any n , while $dJ_{ii}/d\alpha_2$ ($i = 1, 2$) vanishes at least in one point for all n . This vanishing derivative is important in the context of stability of the periodic orbits and their bifurcations.

§3. Stability of the J_{22} symmetric periodic orbits: formation of rings

In order to consider the stability of the periodic orbits, we shall begin by analyzing those of the outer disc. As noticed by Hénon⁹⁾ for the restricted three-body problem, the stability properties are related to the value of $dJ_{ii}/d\alpha_i$. In fact, a saddle-center bifurcation scenario associated with the maxima and minima of the curves J_{ii} is responsible for the appearance or destruction of the symmetric periodic orbits for the rotating one-disc billiard.^{6), 10)} Therefore, just below each of the maxima or above each of the minima, there exists an interval of Jacobi integrals where one of the corresponding periodic orbits is stable and surrounded by invariant tori; the other fixed point corresponds to a hyperbolic one whose manifolds build an incomplete horseshoe. These maxima and minima are defined by the condition $dJ_{22}/d\alpha_2 = 0$, which can be rewritten as

$$\tan \alpha^* = \frac{2}{(2n+1)\pi - 2\alpha^*} \pm 1. \quad (3.1)$$

This equation defines for a given n the angles $\alpha_2 = \alpha^*$ where the maxima or minima are found, independently of the specific geometrical parameters considered in the model.

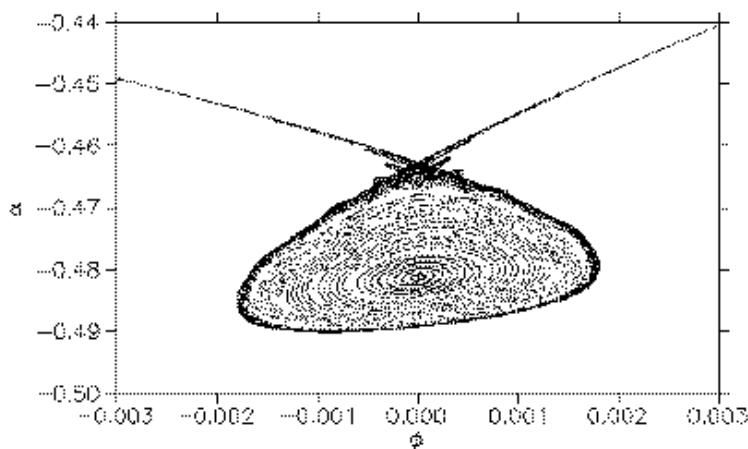


Fig. 3. Poincaré surface of section showing a stability island ($J = 1.173$). An ensemble of initial conditions started within this stable region will not escape along scattering trajectories.

In Fig. 3 we present the surface of section for the one-disc rotating scattering billiard for $J_{22} = 1.173$, illustrating the regions of stable motion just described. We have chosen the parameters $R_2 = 3$ and $d_2 = 1$ ($\omega = 1$). The surface of section is defined here by the angles α and ϕ . The latter angle characterizes the position of the collision point on the disc.⁶⁾ There are only two fixed points of period one for this value of the Jacobi integral, which correspond to the periodic orbits near the maximum of the $n = 0$ case (see Fig. 2a). One of these fixed points is stable and

appears surrounded by KAM tori. These KAM tori as well as some regions of chaotic motion are bounded dynamically by the invariant manifolds of the partner hyperbolic fixed point. This implies that, even-though the problem is actually a scattering billiard system with a (hard disc) repulsive potential, there exist initial conditions of stable motion which are trapped by successive collisions with the rotating disc.

The KAM tori shown in Fig. 3 are robust and persist under small changes of the Jacobi integral. For larger changes, bifurcations occur but those of the central elliptic point are at first hidden from scattering trajectories by surviving KAM surfaces. This scenario persists under variations of the Jacobi integral until the formal parameter describing the (local) binary horseshoes development attains the value $2/3$ (see Refs. 11), 12) for details), which indeed defines a continuous interval for J where bounded trajectories exist. As we further change the value of the Jacobi integral we reach stages of the incomplete horseshoe that correspond to hyperbolic structures and others where very small stable islands reappear.

In the present context, we shall concentrate on a set of initial conditions that remain trapped under small perturbations. That is, we consider all those initial conditions which belong to a bounded region, without considering if their actual motion is periodic, quasi-periodic or chaotic. This set of initial condition has strictly positive measure, i.e. there is a non-zero probability for generic initial conditions of being within this set¹³⁾. Note that this situation is essentially different from the hyperbolic one, where at most a Cantor set of measure zero is trapped whether the horseshoe is complete or incomplete¹¹⁾.

We shall stress that, for a generic ensemble of initial conditions defined on the whole phase space, which is not restricted to a fixed Jacobi integral shell, most initial conditions will escape to infinity. However, particles whose initial conditions are chosen close enough to the stable periodic orbits and within the bounded regions described above, will be trapped by collisions with the disc. This observation is fundamental for understanding the occurrence of patterns or *rings* formed by infinitely many particles whose initial conditions are started precisely on dynamically bounded regions found in distinct intervals of the Jacobi integral.

In Fig. 4a we present in the sidereal frame the ring formed by an ensemble of initial conditions whose Jacobi integral belongs to a neighborhood of the minima of J_{22} for $n = 1$. Figure 4b displays the set of rings formed by the minima of J_{22} for $n = 1, 2$. In these figures, each point represents at a given time the position of a single particle which moves on a rectilinear trajectory. These particles have collisions with the disc at different times and in any point along the disc's circular orbit (sidereal frame). The loop shown in Figs. 4b is associated with the $n = 2$ ring. It results from particles on different rectilinear trajectories (with approximately the same Jacobi integral) which come closer during some interval of time. In general, there will be n loops associated with the retrograde ($\alpha_2 < 0$) J_{22} symmetric periodic orbits, while there will be $n - 1$ loops for the corresponding prograde orbits ($\alpha_2 > 0$). The isolated points in both figures correspond to initial conditions, that were not located in the bounded regions, but have not escaped after the time at which the computation was cut. Another remarkable aspect of Figs. 4 is that instead of viewing it as a stroboscopic map of many non-interacting particles in the sidereal frame, we

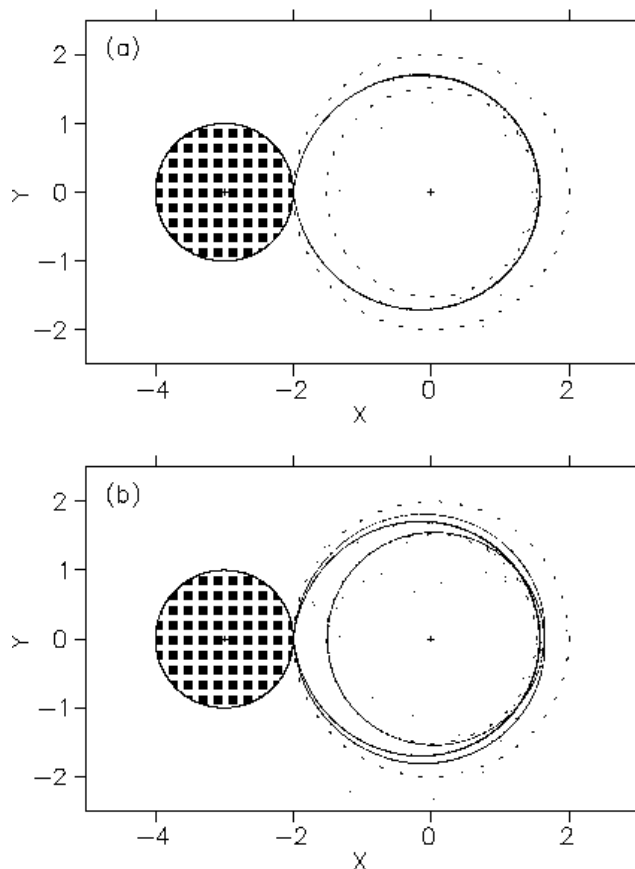


Fig. 4. Examples of rings in the sidereal frame associated with the outer disc (shaded regions) for the rotating two-disc scattering problem. The parameters of the inner disc are defined to filter all but the rings shown (the parameters of the outer disc are the same as in Fig. 2). (a) Ring corresponding to the $n = 1$ minimum of the J_{22} curves ($R_1 = 1$, $d_1 = 0.52$); (b) corresponding rings for the $n = 1, 2$ minima ($R_1 = 0.96$, $d_1 = 0.52$). The dashed circles correspond to the inner and outer circles defined by the orbits of the outer and inner discs, respectively.

could think of the narrow bands of points as curves, and would obtain a picture of the elliptic central orbits of the stable islands of a single particle in the synodic frame.

We can ask about possible structure in the rings of our model. First of all, different rings may coexist, as we see in Fig. 4b. Their particles move at different speeds, and thus probably are not good candidates for the braids of planetary rings. However, we could expect them to be important, say in a semi-classical context, where we might find interference between two structures. Another type of structure is related to the period-doubling scenario, that happens as the elliptic orbit bifurcates. There, several strands of particles may be intertwined, while they move roughly at the same speed and maybe confined to a single ring by surviving exterior KAM surfaces. The details of the structures found may be quite complicated and will depend on the dynamics. Yet the very fact that such structures generically appear

is significant.

§4. Stability of the rings for the two-disc model

It is important to stress that the appearance of rings as described above requires one rotating disc only. Indeed infinitely many rings exist with ever increasing numbers of loops, which give rise to arbitrarily complicated structures in wide rings. For the full two disc rotating billiard such rings may appear associated with either one of the discs. In what follows we shall not consider the rings associated with the inner disc only, as we are interested in narrow rings between the two discs that are actually influenced by both of them. We shall thus concentrate on the periodic orbits associated with the outer disc, which may or may not have collisions with the inner one. The characteristic curves for the J_{22} and J_{12} symmetric periodic orbits are shown in Fig. 5 for angles $\beta = 0, \pi$ between the discs; the size effects of the inner disc are taken into account ($R_1 = 1.8$, $d_1 = 0.1$).

As mentioned in Section 2, the J_{12} (Fig. 2a) characteristic curves for families that bounce on both discs decrease monotonically as a function of α_2 . In fact, it can be shown that $dJ_{12}/d\alpha_2$ is strictly non-zero. Therefore, these families of periodic orbits do not undergo any bifurcation as a function of the Jacobi integral. Yet it was near the bifurcations that we found stable islands for the one-disc orbits. Indeed we do not find any for the two-disc orbits J_{12} and, as we shall see in the next section, there is no reason to expect any. By consequence, these families of periodic orbits do not contribute to the formation of rings. Nevertheless, their presence is intimately related to the disappearance of orbits of the J_{22} type, as can readily be inferred from Fig. 5. This erosion due to the inner disc will in general eliminate some of the extremal points of the characteristic curves; if a sufficiently large interval near the extrema is cut the stable island will no longer exist. To see this, consider initial conditions belonging to some symmetric periodic orbits J_{22} . Depending on the geometrical parameters of inner disc, some of these orbits may collide with the inner disc. The cusps displayed by the J_{12} families indicate the tangent collisions with the inner disc (see Figs. 2 and 5). The particles that collide with the inner disc will loose the needed correlations to build a symmetric periodic orbit, so after a few more collisions these particles will typically escape.

A gap in the characteristic curves for the J_{22} orbits thus occurs. Its location will depend on the angle and radial distance at which the two discs are positioned, while its size will depend on the radius of the inner disc. To illustrate this more clearly Fig. 6 shows the J_{22} curves only, for some angles β .

If the inner disc moves at a speed that is incommensurable with the outer one we find a vastly simplified picture: It will sweep all periodic orbits that penetrate the circle described by the outer edge of this disc. This erosion mechanism is governed in our model by α_2^{\max} compared to the solutions of Eq. 3.1 (which are independent of the geometrical parameters of the model). Its typical time scale depends on the size of the inner disc and the time dependence of β . The absolute value of the solutions of Eq. (3.1) converges to $\pi/4$ as $n \rightarrow \infty$, from above for prograde and from below for the retrograde orbits. This arrangement of the solutions implies that the lowest

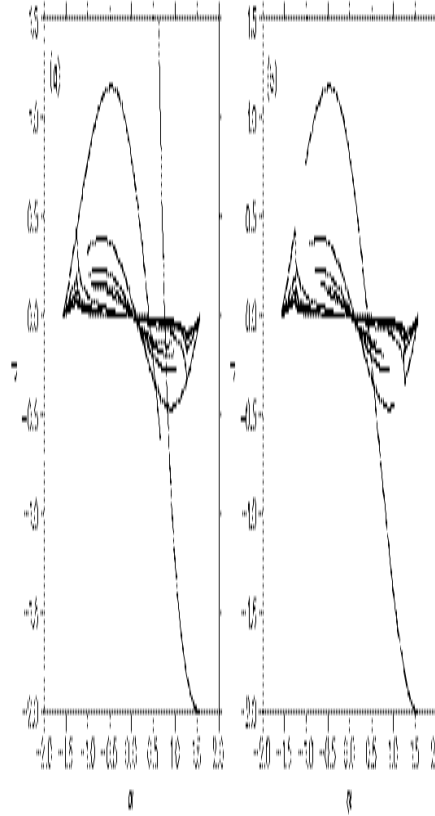


Fig. 5. Symmetric periodic orbits for the two-disc model including the finite size effects of the inner disc. (a) $\beta = 0$; (b) $\beta = \pi$.

prograde solutions will be the first to survive, if we start with very small radial distance of the discs and gradually increase it.⁷⁾ Note though that the $n = 0$ curve in Figs. 2 has no stable prograde periodic orbits in this model (it corresponds to a whispering gallery orbit). If we want to restrict our considerations to a few rings we are forced to look at the prograde part of the branches for the lowest n starting from $n = 1$. Indeed, in Fig. 4a we see a single prograde ring with a circle drawn exactly in the path of the outer edge of the inner disc that ejects all particles that might be associated with other values of n or with retrograde motion. In Fig. 4b the prograde rings for $n = 1$ and $n = 2$ survive. These figures were generated by the dynamics with two discs present and somewhat scattered initial conditions. The points that are not on the rings had unstable initial conditions and are on outgoing scattering

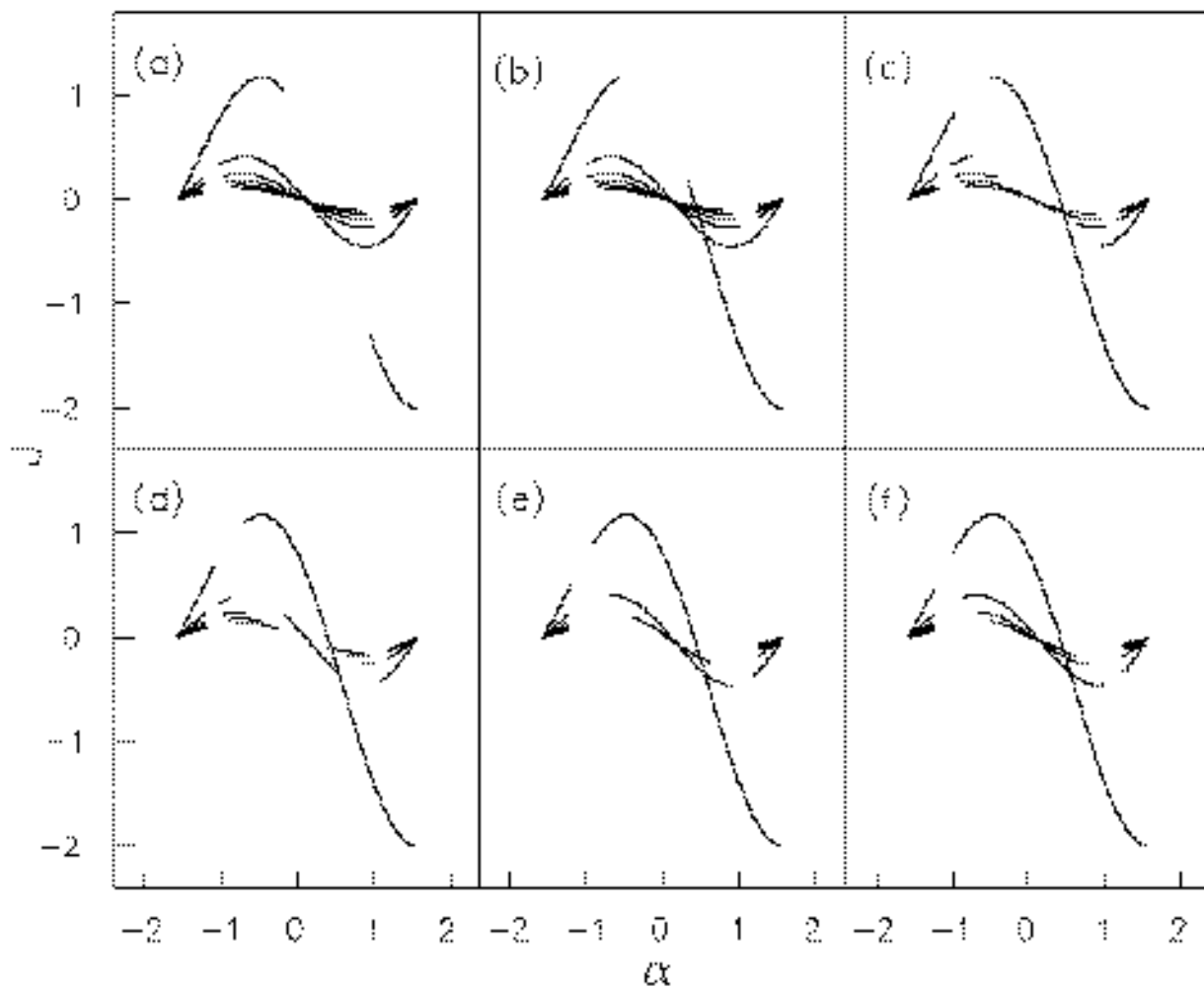


Fig. 6. Periodic orbits of the J_{22} family ($n = 0, 1, \dots, 4$) for different values of β . (a) $\beta = 5^\circ$, (b) $\beta = 10^\circ$, (c) $\beta = 30^\circ$, (d) $\beta = 45^\circ$, (e) $\beta = 90^\circ$ and (f) $\beta = 135^\circ$.

orbits, but have not yet left the domain of the figure for the time we integrated the equations of motion.

Note that for different frequencies of the two discs the Jacobi integral is no longer an invariant of the entire system, but in the subspace of phase space where our rings live the integral is still conserved.

§5. Genericity of the scheme and its importance for planetary rings and other applications

If we consider Fig. 2a, we see that symmetric periodic orbits of the J_{22} family only exist between an upper and a lower limit for the Jacobi integral J . The stable islands discussed in the previous sections are a consequence of the generic scenario of saddle-center bifurcations. Thus we expect and indeed find at each branch point an elliptic and a hyperbolic periodic orbit. The elliptic one will then turn into an inverse hyperbolic fixed point via the period doubling route.

How generic is this scenario? The presence of the cusp where two-disc orbits and one disc orbits touch in their characteristic curves is certainly non-generic. Thus the instability of the two-disc orbits and the behaviour near that cusp will have to be discussed in each particular case with care. As long as the cusp is not too near the maximum of the generic curves, the saddle-center bifurcation will proceed generically. Cases where the two are close will need special attention.

If we consider potential mountains instead of hard discs, we will find an alternative generic scenario; the sudden bifurcation¹⁴⁾ associated with the top of the mountains. Yet, the scenario we discuss will still exist for trajectories that do never reach the region of phase space where the sudden bifurcation occurs.

The question becomes more involved if we consider attractive potentials. In view of the origin of this problem in planetary rings we shall focus on the $1/r$ case. We know from Hénon's work¹⁵⁾ and some of our own¹⁶⁾ that the so called consecutive collision orbits govern the entire chaotic saddle for the restricted three-body problem with small mass parameter μ , i.e. small ratio between the masses of the lighter and heavier binary components. These collision orbits are a good approximation both for the $1/r$ problem with vanishing mass parameter and for the hard disc problem with vanishing disc radius. In fact, they coincide for vanishing disc radius, when we consider an overall central $1/r$ attractive potential. Therefore, the two problems are not quite as distinct as one might believe.

The main difference is that these so called collision orbits may extend both outward and inward from a moonlet. If we consider the orbits that extend inward from the outer moonlet, we are in a situation quite analogous to the one we found for the discs, and the inner moonlet will sweep the ones that reach inside its orbit. If this last observation is to be effective in finite time, we have to pass to finite but small μ . For the orbits that reach outward from a moonlet, the two will interchange their role. The outer one will sweep the collision orbits of the inner moonlet that reach outward beyond the orbit of the outer one. In Fig. 7 we illustrate, in the sidereal frame, both situations with one collision orbit that intersects the orbit of the other moonlet and another that does not. Clearly the zero mass approximation gives no information about stability. Calculations to shed light on this aspect are under way.

Thus, despite of the caveats we pronounced initially, the mechanism we discovered may well be of relevance for the structure of planetary rings. In particular due to the generic nature of the elliptic regions, these are expected to be more stable against perturbations than the orbits of a super integrable system, and could thus be

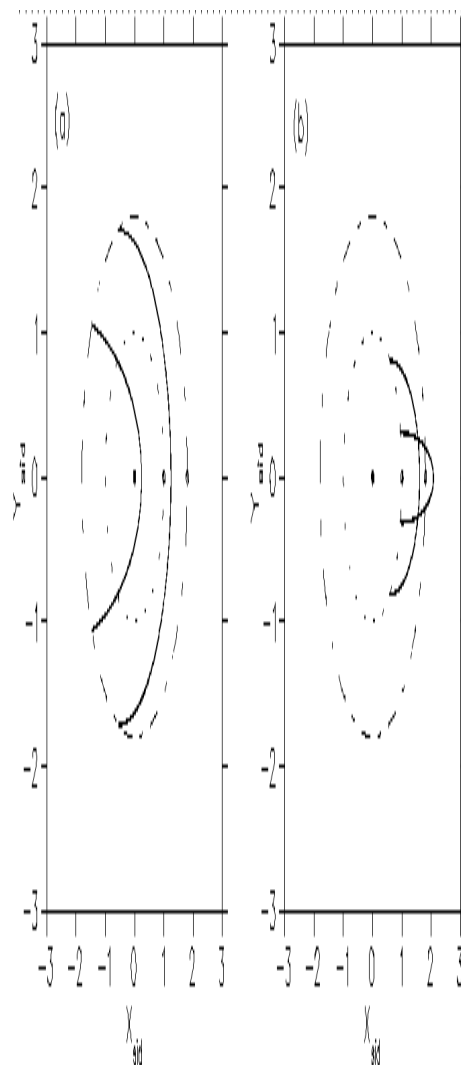


Fig. 7. Examples of consecutive collision orbits in the sidereal frame generated by collisions with (a) the outer moonlet and, (b) the inner one. The dotted and dashed circles represent the circular trajectories of the moonlets. Note that in each picture one of the orbits will be swept by the other moonlet.

associated to a prediction that older ring structures would tend to consist of narrow rings.

Returning to the more general aspects, the genericity of the occurrence, which we suspect also for attractive potentials, could imply importance in other systems such as rotating molecules or high spin states in deformed nuclei in the framework of semi-classical theory. In this case, the mean field potential can be assumed to rotate stiffly and the survival of orbits may depend on the detailed geometry we face. To simplify matters in this situation, the Jacobi integral is a constant of the motion. This may be of interest when the adiabatic or Born-Oppenheimer approximation

breaks down. In nuclei this can well occur for high spin states; the deviations of level statistics from the compound nucleus prediction observed for such systems¹⁸⁾ was attributed to a harmonic component simulated by the centrifugal force, but it is worthwhile to consider the effect of integrable islands of the type mentioned. In molecules it seems less obvious to find a place for non-adiabatic effects, yet Rydberg molecules or molecules with exotic leptons may provide interesting examples.

Acknowledgments

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